# Polarization Considerations

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## Abstract

## Introduction

As light passes through a optical system the reflections and refractions will in general change the polarization state of the light. If we assume that all of the materials in the thin film coatings and substrate are isotropic and homogeneous then calculating the amount of "instrumental" polarization is a relatively straight forward task. In the following sections we will present all of the steps required to perform a "polarization ray trace" calculation for a single ray and monochromatic and hence polarized light. The thin film portion of the calculation is also shown. The reason for explicitly showing the thin film equations is that there are sign conventions imposed on the boundary value equations by the orientation and handedness of the various coordinate frames which are attached to the geometric rays.

The attenuation of light through a optical system, is relatively simple, and requires at the very least a lens (average) reflectivity or transmissivity. Determining the polarization sensitivity of a optical system is still relatively straight forward requiring at least a knowledge of the behavior of the "s" and "p" components at each interface for the chief ray. Determining the thin film induced aberrations of a optical system are somewhat more demanding. Questions about the arithmetic sign of the phase factors and how this relates to the overall "OPD" of a ray are ubiquitous. Many rays are required to construct a wavefront. Thin film codes which modify the OPD's of rays are a requirement for this last mentioned computation. This requires a consistent scheme of coordinate frames and sign conventions and is probably the most demanding task of a polarization ray trace.

Only the electric field will used in the discussion. This is not a restriction as the Stokes parameters are functions of the electric field. The following does not attempt to explain, but only to present all of the required concepts and formulas.

## Maxwell's Equations in a Conducting Medium

It is very convient simply to state that in an isotropic medium with

$$\epsilon = \text{dielectric constant}$$
$$\mu = \text{permeability and}$$
$$\sigma = \text{conductivity}$$

we have

$$\vec{\nabla} \times \vec{H} - \frac{\epsilon}{c}\frac{\partial \vec{E}}{\partial t} - \frac{4\pi\sigma}{c}\vec{E} = 0 \tag{1}$$

$$\vec{\nabla} \times \vec{E} + \frac{\mu}{c}\frac{\partial \vec{H}}{\partial t} = 0 \tag{2}$$

$$\vec{\nabla} \cdot \mu\vec{H} = 0. \tag{3}$$

$$\vec{\nabla} \cdot \epsilon\vec{E} = 0 \tag{4}$$

## Wave Equation

Subject:   Polarization Ray Trace Equations

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The following does not attempt to explain, but only to present all of the required concepts and formulas.

## Maxwell's Equations in a Conducting Medium

In an isotropic medium with

$\epsilon$ = dielectric constant
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then

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$$\nabla \cdot \mu\vec{H} = 0. \tag{4}$$

## Wave Equation

If the electric field is given by

$$\vec{E}(\vec{r}, t) = \vec{E}e^{i(-\omega t + \frac{2\pi N}{\lambda} \hat{k}\cdot\vec{r})} \tag{5}$$

and the row vector by

$$\vec{E}^{+} = \left( E_s^* \ \ E_p^* \right) = \langle E |$$

with $E^*$ denoting complex conjugation and $E^+$ Hermitian conjugation.
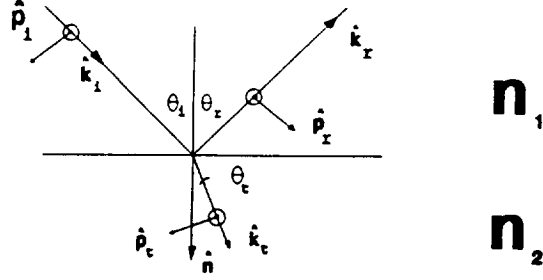
## Local Ray Coordinate Frames

Figure 2: Local Coordinate Frames.

At each interface we have a incident ray $\hat{k}_i$, a reflected ray $\hat{k}_r$ and a transmitted ray $\hat{k}_t$ and a surface normal $\hat{n}$, which is oriented such that the scalar product $\hat{n} \cdot \hat{k}_i > 0$. The local s and p directions for each of these rays is given by

$$\hat{s}_i = \frac{\hat{n} \times \hat{k}_i}{|\hat{n} \times \hat{k}_i|} \tag{11}$$

and with

$$\hat{s}_r = \hat{s}_t = \hat{s}_i$$

and

$$\hat{p}_i = \hat{k}_i \times \hat{s}_i$$

$$\hat{p}_r = \hat{k}_r \times \hat{s}_i \tag{12}$$

$$\hat{p}_t = \hat{k}_t \times \hat{s}_i$$

where $\hat{s}_i$ is perpendicular to the plane of incidence and tangent to

$$\begin{pmatrix} E'_{s1} \\ E'_{p1} \end{pmatrix} = \begin{pmatrix} \hat{s}'_1 \cdot \hat{s}_1 & \hat{s}'_1 \cdot \hat{p}_1 \\ -\hat{s}'_1 \cdot \hat{p}_1 & \hat{s}'_1 \cdot \hat{s}_1 \end{pmatrix} \begin{pmatrix} E_{s1} \\ E_{p1} \end{pmatrix} \quad , \tag{15}$$

or in a more compact notation simply by

$$|E'_1\rangle = R_2 |E_1\rangle$$

where the subscript $R_2$ indicates the incident surface.
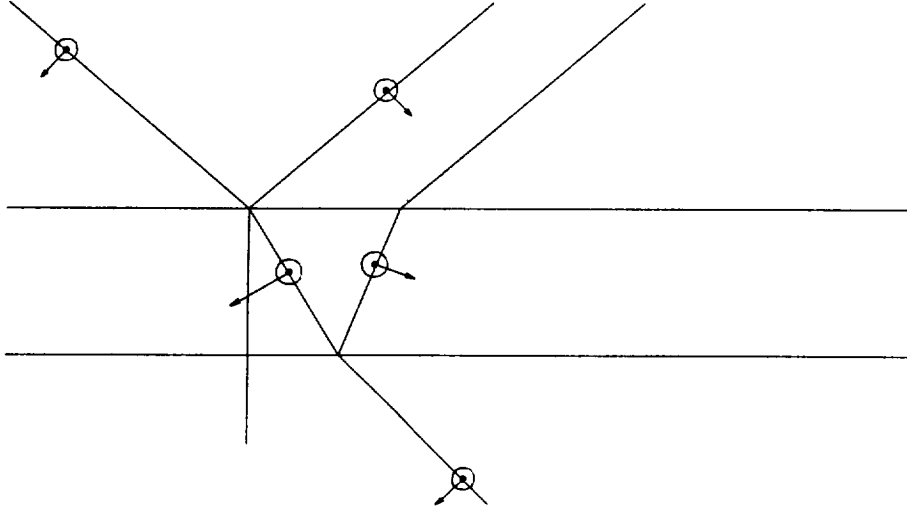
## Boundary Conditions

Figure 4: Two interfaces.

At each interface, within a film stack, the general form of the equations expressing continuity of the tangential components of the E and H fields is similar to the equations at the first interface.

$$(\vec{E}_1 + \vec{E}'_1) \times \hat{n} = (\vec{E}_2 + \vec{E}'_2) \times \hat{n} \tag{16}$$

Comparing equations (19 & 20) and (24 & 25) we see that at each interface we have equations of the form

$$\mathcal{E}_1 + \mathcal{E}_1' = \mathcal{E}_2 + \mathcal{E}_2'$$

$$\zeta_1 (\mathcal{E}_1 - \mathcal{E}_1') = \zeta_2 (\mathcal{E}_2 - \mathcal{E}_2') \quad . \tag{27}$$

With the following substitutions for the <u>s-component</u>

$$\mathcal{E} = E \tag{28}$$

$$\zeta = \frac{N}{\mu} \cos\theta \tag{29}$$

and for the <u>p-component</u>

$$\mathcal{E} = \frac{N}{\mu} E \tag{30}$$

$$\zeta = \frac{\mu}{N} \cos\theta \tag{31}$$

## Multi-Layer Stack Equations

Referring to Figure (5) we have the following equations (for either the s or p components) at the first interface

$$\mathcal{E}_0 + \mathcal{E}_0' = \mathcal{E}_1 + \mathcal{E}_1' \tag{32}$$

$$\zeta_0 (\mathcal{E}_0 - \mathcal{E}_0') = \zeta_1 (\mathcal{E}_1 - \mathcal{E}_1') \quad . \tag{33}$$

At Succeeding (1,1+1) interfaces we have, with

$$\delta_{ij} = \frac{2\pi N_j}{\lambda} d_i \cos\theta_j \tag{34}$$

we have

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## Matrix form of the Equations

Equations (32-38) can be rewritten as

$$\begin{pmatrix} 1 & 1 \\ \zeta_1 & -\zeta_1 \end{pmatrix} \begin{pmatrix} \mathscr{E}_1 \\ \mathscr{E}_1' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \zeta_2 & -\zeta_2 \end{pmatrix} \begin{pmatrix} \mathscr{E}_2 \\ \mathscr{E}_2' \end{pmatrix} \tag{39}$$

$$\begin{pmatrix} e^{-i\delta_{ll}} & e^{i\delta_{ll}} \\ \zeta_l e^{-i\delta_{ll}} & -\zeta e^{i\delta_{ll}} \end{pmatrix} \begin{pmatrix} \mathscr{E}_l \\ \mathscr{E}_l' \end{pmatrix} = \begin{pmatrix} e^{-i\delta_{l+1,l}} & e^{i\delta_{l+1,l}} \\ \zeta_{l+1} e^{-i\delta_{l+1,l}} & -\zeta_{l+1} e^{i\delta_{l+1,l}} \end{pmatrix} \begin{pmatrix} \mathscr{E}_{l+1} \\ \mathscr{E}_{l+1}' \end{pmatrix} \tag{40}$$

$$\begin{pmatrix} e^{-i\delta_m} & e^{i\delta_m} \\ \zeta_m e^{-i\delta_m} & -\zeta e^{i\delta_m} \end{pmatrix} \begin{pmatrix} \mathscr{E}_m \\ \mathscr{E}_m' \end{pmatrix} = \begin{pmatrix} e^{-i\delta_{m+1,m}} & e^{i\delta_{m+1,m}} \\ \zeta_{m+1} e^{-i\delta_{m+1,m}} & -\zeta_{m+1} e^{i\delta_{m+1,m}} \end{pmatrix} \begin{pmatrix} \mathscr{E}_{m+1} \\ \mathscr{E}_{m+1}' \end{pmatrix} \tag{41}$$

or with A and B representing 2 by 2 matrices and "ket" $|e\rangle$ the appropriate column vector we have equations (42)

$$\begin{aligned} A_1 |e_1\rangle &= B_2 |e_2\rangle \\ A_2 |e_2\rangle &= B_3 |e_3\rangle \\ &\vdots \\ A_l |e_l\rangle &= B_{l+1} |e_{l+1}\rangle \\ A_m |e_m\rangle &= |e_{m+1}\rangle \end{aligned} \tag{42}$$

This is equivalent to

$$\begin{aligned} |e_1\rangle &= A_1^{-1} B_2 \; |e_2\rangle \\ |e_2\rangle &= A_2^{-1} B_3 \; |e_3\rangle \\ &\vdots \\ |e_l\rangle &= A_l^{-1} B_{l+1} |e_{l+1}\rangle \\ &\vdots \\ |e_m\rangle &= A_m^{-1} B_{m+1} |e_{m+1}\rangle \end{aligned} \tag{43}$$

and

$$|e_1\rangle = A^{-1} \left( B_2 A_2^{-1} \right) \left( B_3 A_3^{-1} \right) \cdots \left( B_m A_m^{-1} \right) |e_{m+1}\rangle \tag{44}$$

The form of the inverse matrices is

$$A_l^{-1} = \frac{1}{2} \begin{pmatrix} e^{i\delta_{ll}} & \dfrac{1}{\zeta_l} e^{i\delta_{ll}} \\ e^{-i\delta_{ll}} & \dfrac{1}{-\zeta_l} e^{-i\delta_{ll}} \end{pmatrix} \tag{45}$$

## Matrix Form

We can express the reflected and transmitted fields after the $m^{th}$ surface as

$$\begin{pmatrix} E_{m,s} \\ E_{p,m} \end{pmatrix} = \begin{pmatrix} \tau_{s,m} & O \\ O & \tau_{p,m} \end{pmatrix} \begin{pmatrix} E'_{s,m+1} \\ E'_{p,m+1} \end{pmatrix} \tag{53}$$

or

$$\begin{pmatrix} E_s' \\ E_p \end{pmatrix} = \begin{pmatrix} \rho_{s,m} & O \\ O & \rho_{p,m} \end{pmatrix} \begin{pmatrix} E^1_{s,m+1} \\ E^1_{p,m+1} \end{pmatrix} \tag{54}$$

And similarly to equ.(15) and (15.1) this can be written as

$$|E_m\rangle = S_m |E_m^{\wedge}\rangle \tag{55}$$

## Polarization Ray Trace

From equations (15.1) and (55) we see that at each surface the expression

$$|E_m\rangle = S_m R_m |E_{m-1}\rangle \tag{56}$$

connects the electric field before and after each surface. The expression

$$|E_{m+1}\rangle = S_M R_M S_{M-1} R_{M-1} \cdots S_2 R_2 S_1 R_1 |E_0\rangle \tag{57}$$

shows how the incoming plane wave electric field is modified by the intervening surfaces.

## Energy Flux

The magnitude of the energy flux, $J_m$ in a pencil of light, after the $m^{th}$ surface is given by (to within a constant of proportionality)

$$J_m = \frac{n_m \cos \theta_m^A}{n_{m-1} \cos \theta_m^B} \frac{n_{m-1} \cos \theta_{m-1}^A}{n_{m-2} \cos \theta_{m-2}^B} \cdots \frac{n_1 \cos \theta_1^A}{n_0 \cos \theta_1^B} \langle E_m | E_m \rangle \tag{58}$$

$$\tan 2\theta = \frac{(z + z^*)}{z_{11} - z_{22}} \tag{65}$$

or

$$\theta = \frac{1}{2}\tan^{-1}\frac{z + z^*}{z_{11} - z_{22}} \tag{66}$$

and from this we see that the maximum and minimum occur at

$\theta$ and $\theta + \pi/2$.

## Phase Effects

The transmitted intensity is given by

$$J = (s^* \quad p^*)\begin{pmatrix} z_{11} & z \\ z^* & z_{22} \end{pmatrix}\begin{pmatrix} s \\ p \end{pmatrix} \tag{67}$$

where **s** and **p** are complex field quantities. The Z matrix is Hermitian and is the product of the two matrices in equ. (60) where

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix} = S_m R_m \cdots S_2 R_2 S_1 R_1 \tag{68}$$

from equ. (57). Now

$$Z = \begin{pmatrix} z_{11} & z \\ z^* & z_{22} \end{pmatrix} = R_1^{\,t} S_1^{*} R_2^{\,t} S_2^{*} \cdots R_m^{\,t} S_m^{*} S_m R_m \cdots S_2 R_2 S_1 R_1 \tag{69}$$

the t denoting transpose and the * denoting complex conjugation. The general forms of the matrices are

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$$s = \int_0^{\infty} s(\omega) e^{-i\omega t} \, d\omega \qquad (76)$$

$$p = \int_0^{\infty} p(\omega) e^{-i\omega t} \, d\omega \qquad (77)$$

hence

$$sp^*z^* = z^* \int_0^{\infty}\int_0^{\infty} s(\omega) p^*(\omega') e^{-i(\omega-\omega')t} \, d\omega d\omega' \qquad (78)$$

and the time average is given by

$$\frac{1}{\tau}\int_0^{\tau} sp^*z^* = \frac{z^*}{\tau} \int_0^{\infty}\int_0^{\infty} s(\omega) p^*(\omega') \int_0^{\tau} e^{-i(\omega-\omega')t} dt \, d\omega d\omega' \quad . \qquad (79)$$

The real part of the time integral is

$$Re\frac{1}{\tau}\int_0^{\tau} e^{-i(\omega-\omega')t} dt = \frac{1}{\tau(\omega-\omega')} \int_0^{\tau} \cos(\omega-\omega') t \, (\omega-\omega') dt \quad . \qquad (80)$$

From this we see that only when

$$\tau(\omega-\omega') < \frac{\pi}{2} \qquad (81)$$

is the integral different from zero.  For

$$\lambda = 1 \; micron \qquad (82)$$

a band pass filter of $\Delta\lambda \sim 10$ nanometer we get